A Computational Study Of Perspective Cuts

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Semicontinuous Variables

SC variables $\mathbf{x}$ are defined by the following relations:

$$(\underline{\mathbf{x}} - \mathbf{x}^0)z \leq \mathbf{x} - \mathbf{x}^0 \leq (\overline{\mathbf{x}} - \mathbf{x}^0)z,$$
$$z \in \{0, 1\},$$

where z - **indicator variable**.

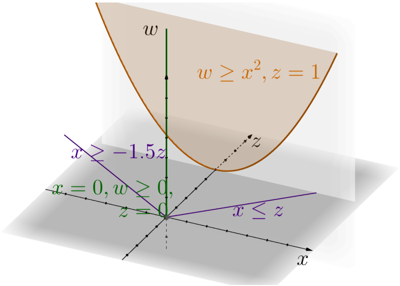
This implies:

$$\mathbf{x} = \mathbf{x}^0 \text{ if } z = 0,$$
$$\mathbf{x} \in [\underline{\mathbf{x}}, \overline{\mathbf{x}}] \text{ if } z = 1.$$

Can be used for describing “on” and “off” states.

Constraints with SC Variables

$$g(\mathbf{x}) \leq w,$$
$$(\underline{\mathbf{x}} - \mathbf{x}^0)z \leq \mathbf{x} - \mathbf{x}^0 \leq (\overline{\mathbf{x}} - \mathbf{x}^0)z,$$
$$z \in \{0, 1\}$$



Example:

$$g(x) = x^2 \leq w,$$
$$-1.5z \leq x \leq z$$

On/off constraints can be written as SC constraints.

Relaxations for SC Constraints

$$g(\mathbf{x}) \leq w,$$
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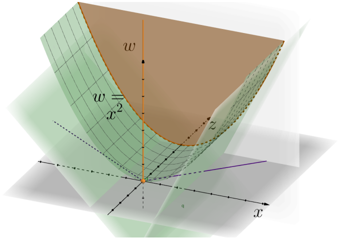
$$\mathbf{z} \in [\mathbf{0}, \mathbf{1}]$$



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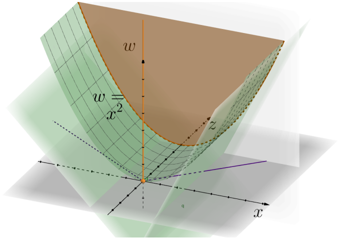
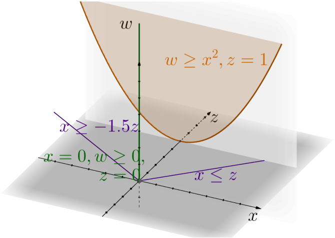


Relaxation can be **improved** with the **use of semicontinuity**!

Disjunctive Formulation



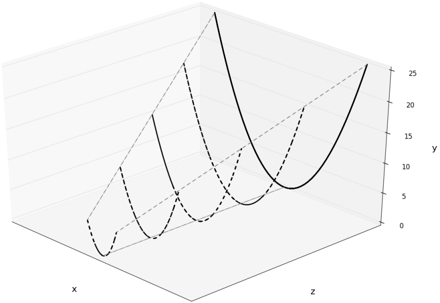
$$S^0 = \{(\mathbf{x}, w, z) \mid \mathbf{x} = \mathbf{x}^0,\ g(\mathbf{x}^0) \leq w,\ \mathbf{z = 0}\},$$
$$S^1 = \{(\mathbf{x}, w, z) \mid \mathbf{x} \in [\underline{\mathbf{x}}, \overline{\mathbf{x}}],\ g(\mathbf{x}) \leq w,\ \mathbf{z = 1}\},$$
$$S = S^0 \cup S^1.$$

Convex hull $=$?

The Perspective Function

$$\tilde{g}(\mathbf{x}, z) = \begin{cases} zg(\frac{\mathbf{x}}{z}) & \text{if } z > 0, \\ +\infty & \text{otherwise} \end{cases}$$

- epi($\tilde{g}$) is a cone generated by epi(g);
- the perspective operator preserves convexity.



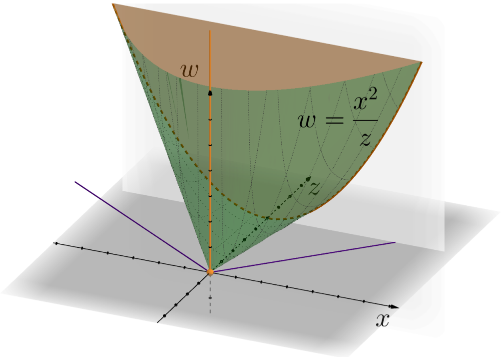
Several dilations of the function $y = x^2$

Perspective Reformulation

If g is **convex**, the perspective function can be used to describe *conv*(S):

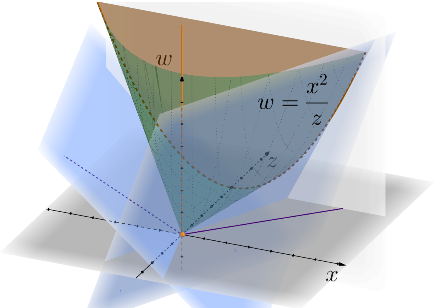
$$cl\{\tilde{g}(\mathbf{x}, z) \leq w\},$$
$$(\underline{\mathbf{x}} - \mathbf{x}^0)z \leq \mathbf{x} - \mathbf{x}^0 \leq (\overline{\mathbf{x}} - \mathbf{x}^0)z.$$

(Need the closure because $\tilde{g}(\mathbf{x}, 0)$ is not well-defined)



Perspective Cuts

Linearise the perspective formulation at $(\mathbf{x}^*, z^*) \Rightarrow$ perspective cuts.



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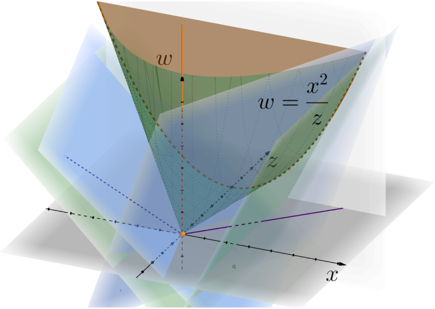


Assuming $\mathbf{x}^0 = g(\mathbf{x}^0) = \mathbf{0}$:

$$\nabla g(\hat{\mathbf{x}})^T \mathbf{x} + (g(\hat{\mathbf{x}}) - \nabla g(\hat{\mathbf{x}})^T \hat{\mathbf{x}}) z \leq w \quad (\hat{\mathbf{x}} = \mathbf{x}^* / z^*)$$

(Frangioni and Gentile, 2006)

Generalised Perspective Cuts

Given **any valid** linear inequality $a\mathbf{x} + b \leq w$ for the set $\{(\mathbf{x}, w) \mid g(\mathbf{x}) \leq w,\ \mathbf{x} \in [\underline{\mathbf{x}}, \overline{\mathbf{x}}]\}$, where $\mathbf{x}$ is semicontinuous, the inequality

$$a\mathbf{x} + b + (a\mathbf{x}^0 + b - g(\mathbf{x}^0))(z - 1) \leq w \qquad (*)$$

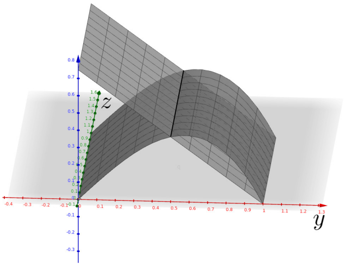
is valid for the disjunctive set

$$\{\mathbf{x} = \mathbf{x}^0, g(\mathbf{x}^0) \leq w, z = 0\} \cup \{\mathbf{x} \in [\underline{\mathbf{x}}, \overline{\mathbf{x}}], g(\mathbf{x}) \leq w, z = 1\}.$$

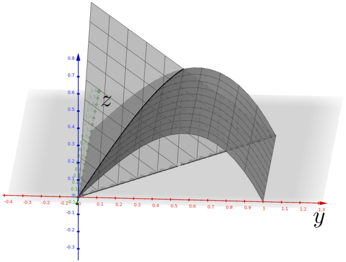
- Cut $(*)$ **does not depend on convexity** of g
- $a\mathbf{x} + b \leq w$ only needs to be valid when $\mathbf{x} \in [\underline{\mathbf{x}}, \overline{\mathbf{x}}]$ (also if $\mathbf{x}^0 \notin [\underline{\mathbf{x}}, \overline{\mathbf{x}}]$)
- If g is convex, cut $(*)$ is equivalent to the perspective cut from [Frangioni, Gentile, 2006]

Example Of Cut Construction

Cut valid for $z = 1$



Cut valid for $z \in \{0, 1\}$



Computations: Setup

- Perspective cuts implemented in SCIP;
- Selected 186 MINLPLib instances that contain suitable structures;
- 4 permutations of each instance + default;
- Time limit one hour.

Computational Results: Summary

Instances with SC structures

All	Convex	Both	Nonconvex
186	89	53	44

Solved and failed instances

	Off	Convex	Full
Solved	741	**764**	759
Limit	175	**154**	**154**
Fails	14	**12**	17

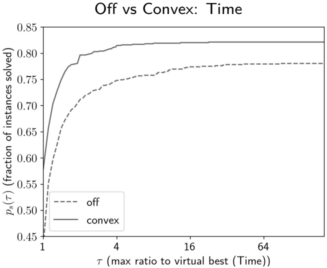
Geometric means of time and nodes

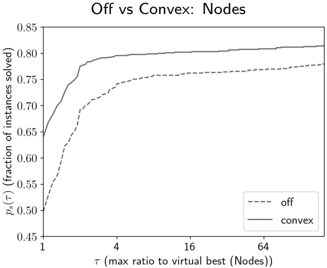
	Off	Convex	Full
Time	13.79	**11.23**	11.27
Relative time	1.00	**0.81**	0.82
Nodes	620	479	**472**
Relative nodes	1.00	0.77	**0.76**

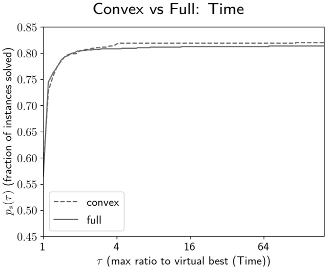
Instances with improvement in root node dual bound

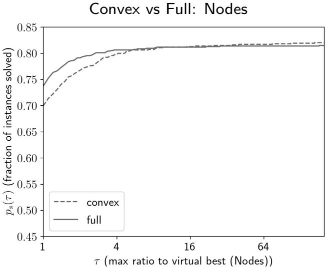
	Off	Convex	Convex	Full
Better by $> 50\%$	16	46	0	31
Better by $5 - 50\%$	25	39	14	11
Same within 5%	584		429	

Computational Results: Performance Profiles







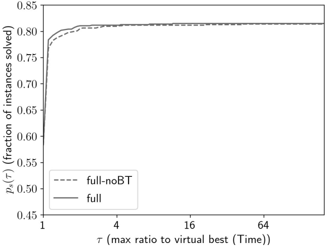


Computational Results: Tighter Bounds

Comparison between Full-noBT and Full

	Fails	Limit	Solved	RootImpr $> 50\%$	Time	Nodes
Full-noBT	16	153	761	4	34.45	2910
Full	17	154	759	25	33.68	2618

Time



Nodes