Strengthening SONC Relaxations with Constraints Derived from Variable Bounds

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Definitions

Consider a **polynomial**

$$f(x) = \sum_{\alpha \in \mathcal{A}} f_\alpha x^\alpha = \sum_{\alpha \in \mathcal{A}} f_\alpha x_1^{\alpha_1} \cdot \ldots \cdot x_n^{\alpha_n}, \; x \in \mathbb{R}^n.$$

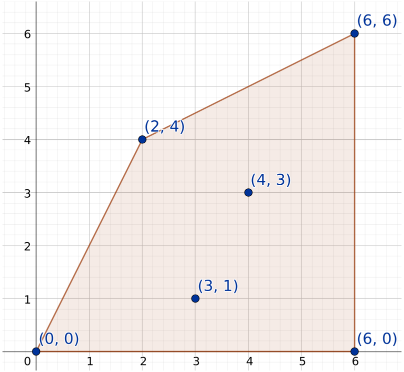
- $\mathcal{A} \subset \mathbb{N}^n$ - support of f
- $New(f) = conv(\{\alpha \in \mathcal{A} \mid f_\alpha \neq 0\})$ - Newton polytope of f (assume $New(f) = conv(\mathcal{A})$)
- $V(\mathcal{A})$ - vertices of $New(f)$
- $\Delta f = \{\alpha \in \mathcal{A} \setminus V(\mathcal{A}) \mid f_\alpha \neq 0$ and $\{f_\alpha < 0$ or $\alpha \notin (2\mathbb{N})^n\}\}$ - non monomial square terms

Assume that **all vertices are monomial squares** (necessary condition for nonnegativity).

Example

$$f(x, y) = 3x^6y^6 + 2x^6 - x^4y^3 - x^3y + 5x^2y^4 - 5$$

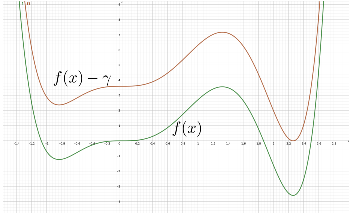
- The figure shows $New(f)$
- $\mathcal{A} = \{(6,6),(6,0),(4,3),(3,1),(2,4),(0,0)\}$
- 4 vertices: (6,6), (6,0), (2,4), (0,0)
- 2 points in the interior: (4,3), (3,1)



Finding Dual Bounds via Nonnegativity Certificates

$$f(x) - \gamma \geq 0 \;\Leftrightarrow\; f(x) \geq \gamma$$

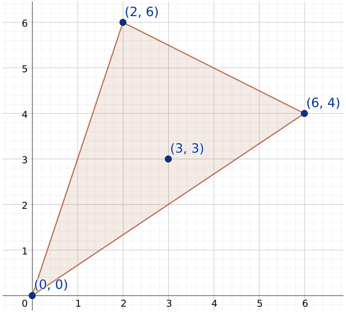
Can find dual bound by maximising γ subject to $f(x) - \gamma \geq 0$.



Circuit Polynomials

$$f(x) = \sum_{\alpha \in V(\mathcal{A})} f_\alpha x^\alpha + f_\beta x^\beta$$

- $\alpha \in V(\mathcal{A})$ are affinely independent.
- β can be written as convex combination of α:
  $\beta = \sum_{\alpha \in V(\mathcal{A})} \lambda_\alpha \alpha, \;\; \lambda_\alpha \geq 0, \;\; \sum_{\alpha \in V(\mathcal{A})} \lambda_\alpha = 1.$



Nonnegativity of circuit polynomials:

- **Circuit number** $\Theta_f = \prod_{\alpha \in V(\mathcal{A})} \left(\frac{f_\alpha}{\lambda_\alpha}\right)^{\lambda_\alpha}$.
- $f(x) \geq 0 \; \forall x$ if and only if:
  $\beta \notin (2\mathbb{N})^n$ and $|f_\beta| \leq \Theta_f$ or
  $\beta \in (2\mathbb{N})^n$ and $f_\beta \geq -\Theta_f$

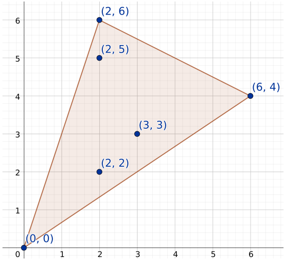
ST-Polynomials

$$f(x) = \sum_{\alpha \in V(\mathcal{A})} f_\alpha x^\alpha + \sum_{\beta \in \Delta f} f_\beta x^\beta$$

is an ST-polynomial if:

- $\alpha \in V(A)$ are affinely independent and
- every $\beta \in \Delta f$ can be written as convex combination of α: $\beta = \sum_{\alpha \in V(\mathcal{A})} \lambda_\alpha \alpha$, $\lambda_\alpha \geq 0$, $\sum_{\alpha \in V(\mathcal{A})} \lambda_\alpha = 1$.

ST-polynomials are similar to circuit polynomials, except that there are several "inner" terms.



Nonnegativity of ST-Polynomials

Write the constant term separately:

$$f(x) = \sum_{\alpha \in V(\mathcal{A}) \setminus 0} f_\alpha x^\alpha + f_{\alpha(0)} + \sum_{\beta \in \Delta f} f_\beta x^\beta, \ \alpha(0) = 0$$

$f(x)$ is **nonnegative** if for every $(\beta, \alpha) \in \Delta f \times V(\mathcal{A})$ there exists $a_{\beta,\alpha} \geq 0$ such that:

$$|f_\beta| \leq \prod_{\alpha \in nz(\beta)} \left(\frac{a_{\beta,\alpha}}{\lambda_\alpha^{(\beta)}} \right)^{\lambda_\alpha^\beta},$$

$$f_\alpha \geq \sum_{\beta \in \Delta f} a_{\beta,\alpha}.$$

This corresponds to a SONC decomposition:

$$f(x) = \sum_{\beta \in \Delta f} \left(\sum_{\alpha \in nz(\beta)} a_{\beta,\alpha} x^\alpha + f_\beta x^\beta \right)$$

Nonnegativity of ST-Polynomials: Reformulation

Rewrite the conditions for nonnegativity of $f(x) - \gamma$ by:

- grouping together everything **not** involving any $a_{\beta,0}$,
- using the left out conditions to formulate a lower bound on $f_0 - \gamma$:

$$\lambda_\alpha^{(\beta)} > 0 \iff a_{\beta,\alpha} > 0 \qquad \leftarrow \text{nonzero coefficients correspond to nonzero } \lambda$$

$$|f_\beta| \leq \prod_{\alpha \in nz(\beta) \setminus 0} \left(\frac{a_{\beta,\alpha}}{\lambda_\alpha^{(\beta)}} \right)^{\lambda_\alpha^{(\beta)}}, \ \forall \beta \text{ with } \lambda_0^\beta = 0 \qquad \leftarrow \text{circuit number condition}$$

$$f_\alpha \geq \sum_{\beta \in \Delta f} a_{\beta,\alpha}, \ \forall \alpha \in V(\mathcal{A}) \setminus 0 \qquad \leftarrow \text{cannot exceed original value of coefficients}$$

$$f_0 - \gamma \geq \sum_{\beta \in \Delta f, 0 \in nz(\beta)} \lambda_0^{(\beta)} |f_\beta|^{1/\lambda_0^{(\beta)}} \prod_{\alpha \in nz(\beta) \setminus 0} \left(\frac{\lambda_\alpha^{(\beta)}}{a_{\beta,\alpha}} \right)^{\lambda_\alpha^{(\beta)}/\lambda_0^{(\beta)}} \qquad \leftarrow \text{above two conditions for } \alpha = 0$$

Finding Lower Bounds on ST-Polynomials

Optimisation problem directly built upon reformulated nonnegativity conditions:

$$\min \sum_{\beta \in \Delta f, 0 \in nz(\beta)} \lambda_0^{(\beta)} |f_\beta|^{1/\lambda_0^{(\beta)}} \prod_{\alpha \in nz(\beta) \setminus 0} \left(\frac{\lambda_\alpha^{(\beta)}}{a_{\beta,\alpha}} \right)^{\lambda_\alpha^{(\beta)} / \lambda_0^{(\beta)}}$$

subject to

$$|f_\beta| \leq \prod_{\alpha \in nz(\beta) \setminus 0} \left(\frac{a_{\beta,\alpha}}{\lambda_\alpha^{(\beta)}} \right)^{\lambda_\alpha^{(\beta)}}, \; \forall \beta \text{ with } \lambda_0^\beta = 0$$

$$f_\alpha \geq \sum_{\beta \in \Delta f} a_{\beta,\alpha}, \; \forall \alpha \in V(\mathcal{A}) \setminus 0$$

Constrained Optimisation

Constrained polynomial optimisation problem:

$$
\begin{aligned}
& \mathit{min}\ f(x) \\
& \text{s.t. } g_i(x) \geq 0 \ \forall i = 1, \ldots, m, \\
& x \in \mathbb{R}^n,
\end{aligned}
$$

where f, g_i are polynomials.

Constrained Optimisation: Lagrangian Approach

$$G(x, \mu) = -\sum_{i=0}^{m} \mu_i g_i(x),$$

where $g_0(x) = -f(x)$.
Separate into monomial squares and tail terms:

$$G(x, \mu) = \sum_{\alpha \in V(G)} f_{\alpha,\mu} x^{\alpha} + \sum_{\beta \in \Delta G} f_{\beta,\mu} x^{\beta}$$

Support depends on μ. We will assume it to be the union of exponents existing for any $\mu \geq 0$.

Goal: find $\max_{\mu} \min_{x} G(x, \mu)$.

Lagrangian Approach: Problem Formulation

Optimisation problem over μ and a simultaneously:

$$\min \sum_{i=1}^{m} \mu_i g_{i,\alpha(0)} + \sum_{\beta \in \Delta G, 0 \in nz(\beta)} \lambda_0^{(\beta)} |f_{\beta,\mu}|^{1/\lambda_0^{(\beta)}} \prod_{\alpha \in nz(\beta) \setminus 0} \left(\frac{\lambda_\alpha^{(\beta)}}{a_{\beta,\alpha}} \right)^{\lambda_\alpha^{(\beta)}/\lambda_0^{(\beta)}}$$

subject to

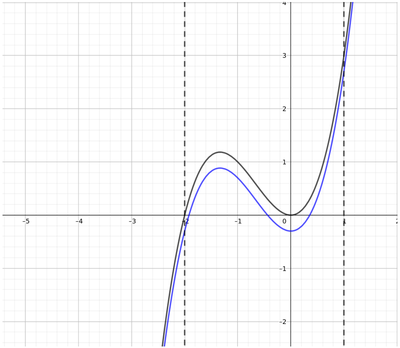
$$|f_{\beta,\mu}| \leq \prod_{\alpha \in nz(\beta) \setminus 0} \left(\frac{a_{\beta,\alpha}}{\lambda_\alpha^{(\beta)}} \right)^{\lambda_\alpha^{(\beta)}}, \ \forall \beta \text{ with } \lambda_0^{\beta} = 0$$

$$f_{\alpha,\mu} \geq \sum_{\beta \in \Delta G} a_{\beta,\alpha}, \ \forall \alpha \in V(G) \setminus 0$$

Can be relaxed into a **geometric program**, which can be transformed into a **convex program**.

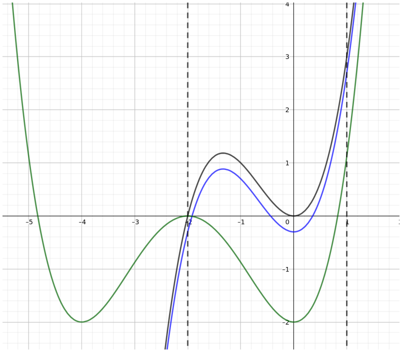
How to Utilise Variable Bounds?

- **Optimisation problem**:
  min $x^3 + 2x^2$ s.t. $-2 \leq x \leq 1$
- **Optimal value** is 0
- $G(x, \mu) = x^3 + 2x^2 - \mu_1(x+2) - \mu_2(1-x)$
- $G(x, \mu)$ **is unbounded** for all $\mu_1, \mu_2 \geq 0$
- The issue is that **variable bounds only add tail terms** to the Lagrangian



Reformulated Bound Constraints

- **Relaxation of the optimisation problem**: min $x^3 + 2x^2$ s.t. $x^4 \leq 32$
- **Optimal value** is 0
- $G(x, \mu) = x^3 + 2x^2 - \mu(16 - x^4)$
- **Lower bound:** $\max_\mu \min_x G(x, \mu) = -2$
- A **SONC certificate can be obtained** proving the lower bound of -2



Choosing Reformulations

Reformulated constraints: $x_i^{\hat{\alpha}} \leq \max\{|l_i|, |u_i|\}^{\hat{\alpha}}$, $\hat{\alpha} \in \hat{\mathcal{A}}$

- **Requirement on $\hat{\mathcal{A}}$**: each $\beta \in \Delta G$ must be representable as a convex combination of exponents from $\mathcal{A} \cup \hat{\mathcal{A}}$
- Each $\beta \in \Delta G$ must be in the interior of the Newton polytope of the extended Lagrangian
- Let $\hat{\mathcal{A}} = \{\hat{\alpha}^{(j)} \mid \hat{\alpha}_i^{(j)} = 0$ if $i \neq j$ and $\hat{\alpha}_i^{(j)} = (n + (n \mod 2)) \cdot \max_{\beta \in \Delta G} \beta, \ \forall j = 1, \dots, n\}$

Computational results

Implemented in SCIP as a relaxator plugin which calls POEM.

Test Run	*Solution Status of the Relaxation*		
	optimal	*decomposition not found*	*did not run/*
Standard SONC	9	333	7
Strengthened SONC	330	0	19

Table: Comparison of the relaxator statuses at root node.